Further Convergence Results for the Weighted Galerkin Method of Numerical Solution of Cauchy-Type Singular Integral Equations

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Abstract. The convergence of the weighted Galerkin method (based on Chebyshev or Jacobi polynomials) for the direct numerical solution of one-dimensional, real Cauchy-type singular integral equations of the first and of the second kind on a finite interval is proved under sufficiently weak continuity assumptions for the kernels and the right-side functions of the integral equations.

1. Introduction. The problem of convergence of the weighted Galerkin method for the direct numerical solution of one-dimensional, real Cauchy-type singular integral equations of the second kind with constant coefficients on a finite interval (called in the sequel simply singular integral equations) [3], [5], [6], [8], [9], [11], [14] will be considered again. The results obtained supplement previous relevant convergence results [9], [11], [14] and, particularly, those of Linz [11] for singular integral equations of the first kind only. The present convergence results make use of the uniform norm like the results of [9], [11] and unlike those of [3], [8], [14].

The convergence of the above method will be proved under sufficiently weak continuity assumptions for the kernels and the right-side functions of the integral equations and will be mainly based on the method used by Linz [11] after a slight modification of this method. This modification consists in basing the convergence results on analogous results for Fredholm integral equations of the second kind [1], [2] and not proceeding directly with the singular integral equations under considera- tion as in [11]. Although this approach, already used in [3], [6], [8], [9], [14], may not be the most direct, it surely is the most economical, yielding a proof of convergence with little effort and on the basis of the well-known theory for Fredholm integral equations of the second kind [1], [2].

2. The Integral Equation. The real singular integral equation to be considered has the form

$$(2.1) \quad a\varphi(x) + \frac{b}{\pi} \int_{-1}^{1} \frac{\varphi(t)}{t-x} dt + \int_{-1}^{1} k(x,t)\varphi(t)\,dt = f(x), \quad -1 < x < 1.$$

The constants a and b are given, as well as the kernel $k(x,t)$ and the right side $f(x)$, which are assumed to be continuous functions. The unknown function is $\varphi(x)$, and

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it is assumed to be possibly unbounded as $x \to \pm 1$. Then [10]

(2.2) $\varphi(x) = w(x)g(x),$

where

(2.3) $w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$

with

(2.4) $-\tan \pi\alpha = \tan \pi\beta = b/a, \qquad -1 < \alpha, \beta < 0.$

The number

(2.5) $\kappa = -(\alpha + \beta) = 1$

is the index [7] of (2.1). Since κ was assumed equal to 1, (2.1) does not possess a unique solution unless supplemented by an additional condition, e.g.,

(2.6) $\int_{-1}^{1} \varphi(t) \, dt = 0.$

The cases when $\kappa = 0$, $\kappa = -1$ or $\kappa = 1$ and 0 is replaced by a constant C in (2.6) are quite similar [3] and will not be considered here.

Because of (2.2), we can rewrite (2.1) and (2.6) as

(2.7) $aw(x)g(x) + \dfrac{b}{\pi} \displaystyle\int_{-1}^{1} w(t) \dfrac{g(t)}{t - x} dt + \int_{-1}^{1} w(t)k(x, t)g(t) \, dt = f(x),$

$$-1 < x < 1,$$

(2.8) $\int_{-1}^{1} w(t)g(t) \, dt = 0.$

We will also use the Fredholm integral equation of the second kind equivalent to (2.7) and (2.8) [7], [9], [10]

(2.9) $g(x) + \displaystyle\int_{-1}^{1} K(x, y)g(y) \, dy = F(x), \qquad -1 \leqslant x \leqslant 1,$

where

(2.10) $K(x, y) = w(y)\left[aw^*(x)k(x, y) - \dfrac{b}{\pi} \displaystyle\int_{-1}^{1} w^*(t) \dfrac{k(t, y)}{t - x} dt \right],$

(2.11) $F(x) = aw^*(x)f(x) - \dfrac{b}{\pi} \displaystyle\int_{-1}^{1} w^*(t) \dfrac{f(t)}{t - x} dt,$

(2.12) $w^*(x) = 1/w(x) = (1 - x)^{-\alpha}(1 + x)^{-\beta}, \qquad 0 < -\alpha, -\beta < 1.$

Before proceeding in our analysis, we will assume that the known functions $k(x, t)$ and $f(x)$ are not only continuous functions (with respect to x and t on $[-1, 1]$), but also that they possess continuous derivatives (at least with respect to x) of orders p_1 and p_2, respectively, on $[-1, 1]$, which are Hölder-continuous [7] with indices μ_1 and μ_2, respectively, that is,

(2.13) $k, f \in C, \quad (\partial/\partial x)^{p_1} k \in H^{\mu_1}, \quad f^{(p_2)} \in H^{\mu_2}.$

The values of $p_{1,2}$ and $\mu_{1,2}$ sufficient for the assurance of the convergence of the weighted Galerkin method for the solution of (2.7) and (2.8) will be specified below.

3. Convergence Results. We consider first the weighted Galerkin method [3], [5] of the numerical solution of (2.7) and (2.8). The existence of a unique solution of the system of linear algebraic equations approximating (2.7) and (2.8) was proved (under sufficiently weak assumptions) in [3]. The convergence of this method was proved in [11], [9] in the special case of singular integral equations of the first kind.

We use the notation $\varphi_i(x)$ for the Jacobi polynomial $P_i^{(\alpha,\beta)}(x)$ of degree i corresponding to α and β, defined in (2.4), and $\varphi_i^*(x)$ for the Jacobi polynomial $P_i^{(-\alpha,-\beta)}(x)$. We assume that the function $g(x)$ has an expansion of the form

$$(3.1) \qquad g(x) = \sum_{i=0}^{\infty} c_i \varphi_i(x),$$

where, because of the orthogonality properties of $\varphi_i(x)$ with respect to $w(x)$ on $[-1, 1]$,

$$(3.2) \qquad c_i = \theta_i^{-1} \int_{-1}^{1} w(t)g(t)\varphi_i(t)\, dt$$

with [13, p. 68]

$$(3.3) \quad \theta_i = \int_{-1}^{1} w(t)\varphi_i^2(t)\, dt = \frac{2^{\alpha+\beta+1}}{2i+\alpha+\beta+1} \frac{\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{\Gamma(i+1)\Gamma(i+\alpha+\beta+1)},$$

where $\Gamma(x)$ denotes the gamma function. Since

$$(3.4) \qquad \Gamma(i+\alpha)/\Gamma(i+\beta) = \mathcal{O}(i^{\alpha-\beta}),$$

we conclude from (3.3) that

$$(3.5) \qquad \theta_i^{-1} \leqslant A_1 i,$$

where A_1 is a positive constant. We will use in the sequel the symbols A_m to denote constants independent of i, but, probably, dependent on the functions used, the variables x and t, etc.

Now we assume that $g \in C^p[-1, 1]$ and further that $g^{(p)} \in H^\mu[-1, 1]$. Then from a corollary of Jackson's theorem [12, pp. 22–23] we conclude that there exists a polynomial $p_i(x)$ of degree less than or equal to i such that

$$(3.6) \qquad \|g - p_i\|_\infty \leqslant A_2 i^{-(p+\mu)},$$

where the uniform norm $\|g\|_\infty$ of a continuous function $g(x)$ is used. Since

$$(3.7) \qquad \int_{-1}^{1} w(t)p_{i-1}(t)\varphi_i(t)\, dt = 0,$$

because of the orthogonality properties of Jacobi polynomials, we find from (3.2)

$$(3.8) \qquad c_i = \theta_i^{-1} \int_{-1}^{1} w(t)\big[g(t) - p_{i-1}(t)\big]\varphi_i(t)\, dt,$$

where $p_{i-1}(t)$ denotes the polynomial of degree $\leqslant i - 1$ of best uniform approximation to $g(t)$. We further obtain from (3.8):

$$(3.9) \qquad |c_i| \leqslant \theta_i^{-1} \|g - p_{i-1}\|_\infty \int_{-1}^{1} w(t)\,|\varphi_i(t)|\, dt.$$

Taking into account that [13, p. 171]

$$(3.10) \qquad \int_{0}^{1} (1-t)^\nu\,|\varphi_i(t)|\, dt \sim i^{-1/2}, \qquad 2\nu > \alpha - 3/2,$$

(the symbol $\sim$ being given the interpretation adopted in [13, p. 1]), we see (since (3.10) holds for $\nu = \alpha$) that

$$(3.11) \qquad\qquad \int_{-1}^{1} w(t) \, |\varphi_i(t)| \, dt \leqslant A_3 i^{-1/2}.$$

On the basis of (3.5), (3.6) and (3.11), (3.9) can be written as

$$(3.12) \qquad\qquad |c_i| \leqslant A_4 i^{-(p+\mu-1/2)}.$$

Hence, the following lemma is proved:

LEMMA 1. *Let $g \in C^p[-1, 1]$ and $g^{(p)} \in H^\mu[-1, 1]$. Then the coefficients c_i of the expansion of g into a series of Jacobi polynomials, determined by (3.2), satisfy (3.12).*

As was shown in [9], the application of the direct weighted Galerkin method (with respect to $w^*(x)$) to (2.7) and (2.8) is completely equivalent to the application of the same method (but with respect to $w(x)$) to (2.9). Moreover, it is also clear that the application of this method to (2.7) and (2.8) yields an approximation $g_n(x)$ of $g(x)$ of the form [1], [2], [3], [5], [9]

$$(3.13) \qquad\qquad g_n(x) = \sum_{i=0}^{n} \gamma_i \varphi_i(x),$$

which is the closed-form solution of (2.7) and (2.8) if $k(x, t)$ and $f(x)$ are similarly approximated by the finite series:

$$(3.14) \qquad\qquad k_n(x, t) = \sum_{k=0}^{n-1} \delta_k(t) \varphi_k^*(x),$$

$$(3.15) \qquad\qquad f_n(x) = \sum_{k=0}^{n-1} \varepsilon_k \varphi_k^*(x),$$

obtained by truncating the corresponding infinite series.

On the basis of (3.13) to (3.15), we see that (2.7) and (2.8) take the form

$$(3.16) \quad aw(x)g_n(x) + \frac{b}{\pi} \int_{-1}^{1} w(t) \frac{g_n(t)}{t-x} \, dt + \int_{-1}^{1} w(t) k_n(x, t) g_n(t) \, dt = f_n(x),$$
$$-1 < x < 1,$$

$$(3.17) \qquad\qquad \int_{-1}^{1} w(t) g_n(t) \, dt = 0,$$

whereas (2.9) to (2.11) can be written as

$$(3.18) \qquad g_n(x) + \int_{-1}^{1} K_n(x, y) g_n(y) \, dy = F_n(x), \qquad -1 \leqslant x \leqslant 1,$$

with

$$(3.19) \qquad K_n(x, y) = w(y) \left[aw^*(x) k_n(x, y) - \frac{b}{\pi} \int_{-1}^{1} w^*(t) \frac{k_n(t, y)}{t-x} \, dt \right],$$

$$(3.20) \qquad F_n(x) = aw^*(x) f_n(x) - \frac{b}{\pi} \int_{-1}^{1} w^*(t) \frac{f_n(t)}{t-x} \, dt$$

by application of the regularization procedure [7], [10] to (3.16) and (3.17).

From now on we will work only with the Fredholm integral equations (2.9) and (3.18) to prove the convergence of the direct weighted Galerkin method applied to (2.7) and (2.8). We assume, to begin with, that (2.7) and (2.8) possess a unique solution. Hence, the same is true with (2.9) and, moreover, with (3.16) and (3.17), as well as with (3.18), for sufficiently large values of n. A large amount of theoretical results for Fredholm integral equations of the second kind is available [1], [2]. We can use, for example, Theorems 4.3, 4.4 or 4.5 of [2, pp. 426–430]. Theorem 4.4, slightly modified and adapted to our notation, states that

THEOREM 1. *Suppose that*

$$(3.21) \qquad \|(I + K)^{-1}\|_\infty^* \|K - K_n\|_\infty^* < 1.$$

Then

$$(3.22) \quad \|g - g_n\|_\infty$$

$$\leqslant \frac{\|(I + K)^{-1}\|_\infty^*}{1 - \|(I + K)^{-1}\|_\infty^* \|K - K_n\|_\infty^*} [\|F - F_n\|_\infty + \|g\|_\infty \|K - K_n\|_\infty^*].$$

In this theorem uniform norms of operators are also used [2, p. 22].

From (3.22) it is evident that the convergence of $g_n(x)$ to $g(x)$ is assured if $\|F - F_n\|_\infty \to 0$ and $\|K - K_n\|_\infty^* \to 0$ for $n \to \infty$. Moreover, if $\|K - K_n\|_\infty^* \to 0$ for $n \to \infty$, (3.21) is also fulfilled under the already made assumption that (2.7) and (2.8) and, equivalently, (2.9) possess a unique solution. Hence $(I + K)^{-1}$ exists.

We now recall that [13, p. 166]

$$(3.23) \qquad \|\varphi_i\|_\infty \sim i^q, \qquad i \to \infty,$$

where

$$(3.24) \qquad q = \max(\alpha, \beta, -\tfrac{1}{2}) = \max(\alpha, \beta)$$

because of (2.4) and (2.5). Hence, we also have

$$(3.25) \qquad \|\varphi_i^*\|_\infty \sim i^{q^*}, \qquad i \to \infty, \qquad q^* = \max(-\alpha, -\beta) = -\min(\alpha, \beta).$$

By taking now into account that [10]

$$(3.26) \qquad aw(x)\varphi_i(x) + \frac{b}{\pi} \int_{-1}^1 w(t)\frac{\varphi_i(t)}{t - x}dt = \frac{b}{2\sin \pi\alpha}\varphi_{i-1}^*(x),$$

$$(3.27) \qquad aw^*(x)\varphi_i^*(x) - \frac{b}{\pi} \int_{-1}^1 w^*(t)\frac{\varphi_i^*(t)}{t - x}dt = -\frac{2b}{\sin \pi\alpha}\varphi_{i+1}(x)$$

and the restriction (2.5), we easily obtain from (3.14), (3.15), (3.19) and (3.20)

$$(3.28) \qquad K_n(x, y) = -\frac{2bw(y)}{\sin \pi\alpha} \sum_{k=0}^{n-1} \delta_k(y)\varphi_{k+1}(x),$$

$$(3.29) \qquad F_n(x) = -\frac{2b}{\sin \pi\alpha} \sum_{k=0}^{n-1} \varepsilon_k\varphi_{k+1}(x).$$

Therefore,

$$(3.30) \qquad K(x, y) - K_n(x, y) = -\frac{2bw(y)}{\sin \pi\alpha} \sum_{k=n}^{\infty} \delta_k(y)\varphi_{k+1}(x),$$

$$(3.31) \qquad F(x) - F_n(x) = -\frac{2b}{\sin \pi \alpha} \sum_{k=n}^{\infty} \varepsilon_k \varphi_{k+1}(x),$$

On the basis of Lemma 1 and (3.23), we see further that

$$(3.32) \qquad \| F - F_n \|_\infty \leqslant A_5 n^{q+3/2-(p_2+\mu_2)},$$

$$(3.33) \qquad \| K - K_n \|_\infty^* \leqslant A_6 n^{q+3/2-(p_1+\mu_1)}.$$

Then from Theorem 1 it follows that:

THEOREM 2. *Suppose that* (2.13) *and* (3.21) *hold and*

$$(3.34) \qquad \gamma = \min(p_1 + \mu_1, p_2 + \mu_2) - (q + 3/2) > 0.$$

Then

$$(3.35) \qquad \| g - g_n \|_\infty \leqslant A_7 n^{-\gamma}$$

for the direct weighted Galerkin method of numerical solution of singular integral equations.

This theorem assures the convergence of the method under the assumptions already made. Consider as an application the case when $\alpha = \beta = -\frac{1}{2}$, $\mu_1 = \mu_2 = 1$, $p = \min(p_1, p_2)$. Then from (3.34) $\gamma = p$, and Theorem 3 of Linz [11] is derived again. An improved convergence rate was obtained in [9] in this particular case.

Consider now in more detail the same special case where $\alpha = \beta = -\frac{1}{2}$ (but independently of the values of $\mu_{1,2}$). Because of (3.14) and (3.15), we obtain

$$(3.36) \qquad \| f - f_n \|_\infty \leqslant A_8 n^{-(p_2+\mu_2)} \log n, \qquad a = 0,$$

$$(3.37) \qquad \| k - k_n \|_\infty^* \leqslant A_9 n^{-(p_1+\mu_1)} \log n, \qquad a = 0,$$

as is clear from the results reported in [12, pp. 22–23, 61].

Then, by working exactly as in [4, p. 288], we conclude, on the basis of (2.10), (2.11), (3.19), (3.20) and (3.36), (3.37), that

$$(3.38) \qquad \| F - F_n \|_\infty \leqslant A_{10} n^{-(p_2+\mu_2)+\varepsilon}, \qquad a = 0,$$

$$(3.39) \qquad \| K - K_n \|_\infty^* \leqslant A_{11} n^{-(p_1+\mu_1)+\varepsilon}, \qquad a = 0,$$

where ε is an arbitrarily small positive constant. These results are stronger than (3.32) and (3.33), respectively, for the special case considered ($a = 0$). The same results lead also directly to:

THEOREM 3. *Suppose that* $a = 0$ *and*

$$(3.40) \qquad \gamma = \min(p_1 + \mu_1, p_2 + \mu_2) - \varepsilon > 0,$$

where ε is an arbitrarily small positive constant. Then Theorem 2 holds true for this value of γ.

Unfortunately, it has not been possible up to now to generalize this result for singular integral equations of the second kind. From Theorems 2 and 3 it follows also that under the assumptions made in this section the direct weighted Galerkin method of numerical solution of singular integral equations converges if $k, f \in H^\mu[-1, 1]$ ($0 < \mu \leqslant 1$) (with respect to the x-variable) in the case of singular integral equations of the first kind, and at least if $(\partial/\partial x)k, f^{(1)} \in H^{\mu^*}[-1, 1]$ ($\frac{1}{2} \leqslant \mu^* \leqslant 1$) in the case of singular integral equations of the second kind.

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